Variational calculus and Poincare-Cartan formalism on supermanifolds

1995 J. Phys. A: Math. Gen. 28 723

(http://iopscience.iop.org/0305-4470/28/3/025)

J. Phys. A: Math. Gen. 28 (1995) 723–734. Printed in the UK

Variational calculus and Poincaré–Cartan formalism on supermanifolds

Roberto Ciancit†, Mauro Francaviglia‡ and Igor Volovich§

† Dipartimento di Matematica, Università di Genova, Italy
‡ Istituto di Fisica Matematica 'J-L Lagrange', Università di Torino, Italy
§ Steklov Mathematical Institute, Academy of Science of the Russian Republic, Moscow, Russia

Received 21 September 1994, in final form 5 December 1994

Abstract. In this work we show how the Poincaré–Cartan form can be used to describe the symmetries of Lagrangian supersymmetric (possibly p-adic) field theories. We show that the existence of the Poincaré–Cartan form in supermanifold theory is ensured only in a relevant class of Lagrangian densities. Moreover, we give an abstract characterization of supersymmetric invariance based on the Poincaré–Cartan form.

1. Introduction

Various supersymmetric models of field theories have been considered (for a review see [1–3]). Unfortunately there is as yet no experimental evidence of supersymmetry in nature, although ideas from supersymmetry have proved very useful in mathematics [2]. The notion of a supermanifold [4–8], which has mainly been developed to deal with supersymmetric theories, gives a convenient approach for discussing an arbitrary field theory, not necessarily a supersymmetric one.

As is well known, in the Feynman path integral approach to quantization one deals with a *classical* action in which fermionic fields are represented by anticommuting fields. Also recall that any gauge theory being quantized by using the BRST procedure should include anticommuting variables (see [9]).

In this article, we develop an approach based on supermanifold theory which is suitable to describe models with anticommuting fields which are not necessarily supersymmetric. This provides us with a framework which is general enough to include such different theories as the usual quantum electrodynamics, the Yukawa interaction, superstrings and supergravity, as well as BRST-quantized gauge theories. Although many models with anticommuting fields have already been considered in great detail in the literature, the purpose of this article is to give an exposition of theories with anticommuting fields with the same degree of generality as occurs in classical mechanics [9–11].

We consider a functional analytic approach to superanalysis and we obtain a unified approach to finite dimensional as well as to infinite dimensional systems. This could be interesting, for example, in view of the known analogy existing between the description of the rigid body motion in finite dimensional mechanics and the infinite dimensional theory of incompressible fluids [12].

Our main results are true also in the field of p-adic numbers (for p-adic superanalysis, quantum mechanics and field theory see [13–15]).

We consider field theories with fields taking values on a supermanifold and arguments on a real or p-adic manifold and we discuss fields which are maps from a supermanifold to another supermanifold. We study the action of superdiffeomorphisms on the Lagrangian density $\mathcal{L}$ and the Poincaré–Cartan form Θ.

Some elements of the theory of the action of the superdiffeomorphism group on a supermanifold field theory can be found in the article [16]; since we use some notation already introduced in that article, we suggest to the reader not familiar with jet-bundle theory to read this article. In the present article we significantly extend the results presented there by studing in great detail the action of the superdiffeomorphism group by introducing a particular system of coordinates called 'covariant coframe coordinates' which will allow us to obtain a simple way to characterize supersymmetric Lagrangian density.

Moreover, by returning to Poincaré–Cartan form theory, we show that Θ exists globally if $\mathcal{L}$ is *geometric* and, in this case, the field equations follow from the Poincaré–Cartan form as in usual field theory (see, e.g., [17]).

As an example, we shall explicitly consider the theory of $N = 1$ supergravity and we shall give the relevant Poincaré–Cartan form.

We remark that the whole discussion will be given in a consequential way and results will be summarized as propositions which, accordingly, will not be followed by 'formal' proofs.

2. Some concepts from real and p-adic supermanifold theory

As is well known, there are several mathematical approaches to supermanifold theory. For our purposes it will be convenient to rely on the definitions in [4]. The main ingredient of a supermanifold theory is a Z_2-graded commutative Banach superalgebra Q over a normed field A [13–18]; for the real Grassmannian case see [19–22]. More precisely, Q is a Z_2-graded commutative Banach algebra $Q = Q_0 \oplus Q_1$ such that $a_i a_j = (-1)^{ij} a_j a_i$ if $a_i \in Q_i, a_j \in Q_j$.

Let us remark that the field A over which Q is a Banach space is assumed to be quite general: it can be $\boldsymbol{R}, \boldsymbol{C}$ or also the field $\boldsymbol{Q}_p$ of p-adic numbers. As a consequence, our theory is also true in the framework of p-adic supermanifold theory [23–25]. Since p-adic supermanifold theory is not very extensively covered in the literature, we present some details.

The (n, m)-dimensional superspace is the A-linear Banach space $V^{n,m} = Q_0^n \times Q_1^m$ endowed with the norm $\|u\| = \sum_{j=1}^{n+m} \|u_j\|$; in the p-adic case we set $\|u\| = \max_{1 \leqslant j \leqslant n+m} \|u_j\|$ in order to obtain a non-Archimedean norm.

A map $f: V^{n,m} \to Q$ is called S-differentiable (or supersmooth) in $x \in V^{n,m}$ if it is differentiable in the Frechét sense [26] and if its Frechét differential $(Df)(x): V^{n,m} \to Q$ in the point x is multiplicative. In this article we require that the multiplicative operator describing the Frechét operator is unique; to this end, we assume that Q admits a trivial Q_1-annihilator ([27]).

A supersmooth map f will be called an S^k map if it is k-times continuously superdifferentiable. The set of superanalytic (abbreviated SA) maps $f: U \subset V^{n,m} \to Q$ is denoted as $S^\omega(U)$. (for the p-adic manifold case see [28] for details).

We now define $SC^k(U)$ to be the set of $S^k(U)$ maps such that $\|D^k f(x)\|$ is bounded for any $x \in U$. If $k \neq \omega$, $SC^k(U)$ is a Banach space with respect to the norm

$$\|f\| = \sum_{i=0,\ldots,k} \sup_{x \in U} \|D^i f(x)\|.$$

An (n, m)-dimensional S^k supermanifold M is a topological manifold with a superdifferential structure. In some cases the algebra Q admits a projection $\varepsilon: Q \to A$, often called the body map. In these cases, under some further technical assumptions [4, 29], one can construct the ordinary n-dimensional *body* manifold M_0 by glueing together all the local projections $\varepsilon\left(x^1, \ldots, \theta^{n+m}\right) = (r^1, \ldots, r^n, 0, \ldots, 0)$.

Let E be a supervector bundle $\pi: E \to M$ with the vector superspace $\mathbf{F}$ as standard fibre ([4]).

Let μ be an S^k *automorphism* of E. Its *first jet extension* $j\mu$ of μ is the *unique* automorphism of JE such that, for any local section $s: U \subset M \to E$, $j\pi \circ j\mu = \beta \circ j\pi$ and $j\mu \circ js \circ \beta^{-1} = j(\mu \circ s\beta^{-1})$ (see [30]).

Finally, consider a one-parameter group $\{\mu_t\}$ $(t \in A)$ of supersmooth automorphisms of E and denote by Y its generator (for p-adic supermanifold theory we assume also that every μ_t is strictly differentiable [23, 24]).

The generator jY of $\{j(\mu_t)\}$ is the *jet extension* of Y. In local fibred coordinates of E, where $x^A = (x^1, \ldots, x^n, \theta^{n+1}, \ldots, \theta^{n+m})$ are coordinates in M and $v^\alpha = (v^1, \ldots v^r, v^{r+1}, \ldots, v^{r+s})$ are coordinates in $\mathbf{F}$ $(\dim \mathbf{F} = (r, s))$, we set

$$Y = e^A(x)\frac{\partial}{\partial x^A} + h^\alpha(x, v)\frac{\partial}{\partial v^\alpha}. \tag{1}$$

By using local fibred coodinates $(x^A, v^\alpha, v_B{}^\alpha)$ in JE, we get

$$jY = e^A(x)\frac{\partial}{\partial x^A} + h^\alpha(x, v)\frac{\partial}{\partial v^\alpha} + \left(\frac{\partial h^\alpha(x, v)}{\partial x^B} + v_B{}^\beta \frac{\partial h^\alpha(x, v)}{\partial v^\beta} - \frac{\partial e^A}{\partial x^B} v_A{}^\alpha\right)\frac{\partial}{\partial v_B^\alpha}. \tag{2}$$

3. Supermanifold Lagrangian theory

In a supermanifold theory there are essentially two possibilities for constructing the action functional; the first one relies on using the Berezin integral to construct a 'top form' in M and defining the action as functional by means of a 'scalar Lagrangian' $\mathcal{L}$ as

$$\int \mathcal{L}\, \mathrm{d}^n x\, \mathrm{d}^m \theta. \tag{3}$$

In the second approach we assume instead that there exists a body manifold $M_0 = \Phi(M)$ of M and consider a set of *local injections* $\{i_\alpha\}$ (see [16] and the following for details).

As usual we denote the 'configuration bundle' over M by E; the standard fibre is a vector superspace F and local coordinates are (x^A, v^r). We also assume that the Lagrangian is a bounded, horizontal and SA n-form of JE with values in Q_0, i.e. $\mathcal{L} \in \operatorname{hor}\Omega^n(JE)$.

Now, for every open subset U_α of M, for every local injection $i_\alpha: V_\alpha \to U_\alpha$ (V_α denotes the body of U_α) and for every section s of E, we define the action integral as

$$A_\alpha(i, s) = \int_{V_\alpha} i^*(js)^*\mathcal{L}. \tag{4}$$

This integral is calculated on M_0 and it is a standard integral of Banach valued functions (see also [7, 18, 31, 32]).

The second approach allows us to build a very rigorous integration theory and a variational calculus in the framework of Banach analysis; in this article we shall rely on this second approach and, for the sake of clarity, we begin by giving some further detail.

If M is an SA supermanifold with body M_0 there always exists an atlas A of M $\{(U_\alpha, \psi_\alpha)\}$ and an atlas A_0 of M_0 $\{(V_\alpha, \xi_\alpha)\}$ such that [16]

(i) $\Phi(U_\alpha) = V_\alpha$ for all α;

(ii) for any α there are analytic embeddings $i: V_\alpha \to U_\alpha$ (local injections) such that $\Phi \cdot i$ is surjective;

(iii) for any couple of local injections $i, k: V_\alpha \to U_\alpha$ there is an SA automorphism χ of M such that $i = \chi \cdot k$;

(iv) for any map i there are maps $f: V_\alpha \to Q_0$ such that $i^* f|_{\partial V_\alpha} = 0$;

(v) if ω is a p-form on $U \subset M, p \leqslant n$ and $i^*\omega = 0$ for all local injections $i: V \to U$ then $\omega = 0$.

Notice that these conditions do not imply, in general, that a global injection of M_0 in M exists.

Now we give a Banach structure to the space of local sections of E by stating that the completed tensor product $B_\alpha = SC^k(U_\alpha) \otimes \mathbf{F}$ is a Banach space.

Now we consider the problem of defining the space of *admissible variations*. For every V_α and every injection i_α the set of admissible variations related to the open subset U_α and to the local injection $i: V_\alpha \to U_\alpha$ is the Banach space $B_{\alpha,i} = \{s \in \Gamma(U_\alpha) | i^* s = 0 \text{ on } \partial V_\alpha\}$. $\Gamma(U_\alpha)$ denotes the set of local sections s defined on $U_\alpha \in M$ with values in $\pi^{-1}(U_\alpha) \in E$.

This space can be regarded as the space of tangent fields $X \in \operatorname{Vert} TE$ such that their components X^r satisfy the equations $(i^* X^r)|_{\partial V_\alpha} = 0$.

Let us consider again the action functional (4). *The following two equations are* equivalent and characterize the critical sections s :

$$DA_\alpha(i, s) \cdot X = 0 \tag{5}$$

$$\int_{V_\alpha} i^* \cdot js^* L_{jX} \mathcal{L} = 0 \tag{6}$$

for all α, all i and all X s.t. $X^r \in B_{\alpha,i}$.

In the following we use coordinates $x^A = (x^i, \theta^\alpha)$ on M $(i = 1, \dots n; \alpha = 1, \dots, n)$ and coordinates v^r on $\mathbf{F}$; (x^A, v^r) are coordinates in E and (x^A, v^r, v_A^r) in JE.

According to this notation the Lagrangian $\mathcal{L}$ is

$$\mathcal{L} = \frac{1}{n!} \sum\nolimits_{A_1 \dots A_n} \mathrm{d}x^{A_1} \dots \mathrm{d}x^{A_n} \mathcal{L}_{A_1 \dots A_n}(v^r, v_B^r)$$

where $A_1, A_2, \dots = 1, \dots, m + n$.

In this framework a particular role is played by the so-called *geometrical* Lagrangian forms. They are the Lagrangians for which there exists forms $\mathcal{L}/v'^r$ which allow us to write

$$h_A^r \frac{\partial \mathcal{L}}{\partial v_A^r} = h^r \frac{\partial \mathcal{L}}{\partial v'^r} \tag{7}$$

where $h^r = dx^A h_A^r$ for any admissible variation h^r. The terms $\mathcal{L}/v_A^r, \mathcal{L}/v'^r$ are related by the equation

$$\frac{\partial \mathcal{L}}{\partial v_A^r} = \mathrm{d}x^A (-1)^{A(A+r)} \frac{\partial \mathcal{L}}{\partial v'^r} \tag{8}$$

which, in turn, can be written as a set of algebraic restrictions on the derivatives of $\mathcal{L}$. Lagrangian forms which are built up by means of connection forms, curvature and torsion forms, by using the wedge exterior product are geometrical ([16]).

Proposition 3.1. Let us assume that $\mathcal{L}$ is a geometrical Lagrangian; a section s is critical iff

$$js^* \frac{\partial \mathcal{L}}{\partial v^r} - \mathrm{d}\,(js)^* \frac{\partial \mathcal{L}}{\partial v'^r} = 0. \tag{9}$$

Proof. For $X \in B_{\alpha,i}$ we put $X = b^r \partial/\partial v^r$ and compute $i^\star(js)^\star L_{jX}\mathcal{L} = i^\star(js)^\star jX \rfloor \mathrm{d}\mathcal{L}$. We have

$$\begin{aligned}\int i^\star(js)^\star jX\rfloor \mathrm{d}\mathcal{L} &= \int i^\star(js)^\star b^r\frac{\partial\mathcal{L}}{\partial v^r} + \int i^\star\frac{\partial}{\partial x^A}\left(s^\star b^r(js)^\star\frac{\partial\mathcal{L}}{\partial v_A^r}\right)\\ &\quad -\int i^\star(s^\star b^r)(-1)^{Ar}\frac{\partial}{\partial x^A}\left((js)^\star\frac{\partial\mathcal{L}}{\partial v_A^r}\right)\\ &= \int i^\star(s^\star b^r)\left((js)^\star\frac{\partial\mathcal{L}}{\partial v^r} - (-1)^{Ar}\frac{\mathrm{d}}{\mathrm{d}x^A}(js)^\star\frac{\partial\mathcal{L}}{\partial v_A^r}\right)\\ &\quad +\int i^\star\frac{\partial}{\partial x^A}\left((s^\star b^r)((js)^\star\,\mathrm{d}x^A\frac{\partial\mathcal{L}}{\partial v'^r}(-1)^{A(A+r)}\right)\\ &= \int i^\star(s^\star b^r)\left((js)^\star\frac{\partial\mathcal{L}}{\partial v^r} - \mathrm{d}(js)^\star\frac{\partial\mathcal{L}}{\partial v'^r}\right) + \int i^\star\,\mathrm{d}\left((s^\star b^r)(js)^\star\frac{\partial\mathcal{L}}{\partial v'^r}\right).\end{aligned} \tag{10}$$

We have reported the explicit calculation, since our hypothesis about $\mathcal{L}$ allows us to relax the so-called 'kinematical constraints' of Wess and Zumino [33]. By using the fact that the local injection i is arbitrary the first term vanishes which, in turn, yields the required field equations since the second term vanishes for the boundary conditions. □

Now we study the Poincaré–Cartan form; it has the following expression

$$\Theta = \mathcal{L} + (\mathrm{d}v^r - \mathrm{d}x^A v_A^r)\ Q_r \tag{11}$$

where the forms $Q_r \in \mathrm{hor}\,\Omega^{n-1}(JE)\otimes F^\star$ are still to be determined.

Now we suppose $\mathcal{L}/x^A = 0$ and compute the differential

$$\mathrm{d}\Theta = \mathrm{d}v^r\frac{\partial\mathcal{L}}{\partial v^r} + \mathrm{d}v_A^r\frac{\partial\mathcal{L}}{\partial v_A^r} - \left(\mathrm{d}v^r - \mathrm{d}x^B v_B^r\right)\,\mathrm{d}Q_r + \mathrm{d}x^A\,\mathrm{d}v_A^r\,Q_r. \tag{12}$$

Considering a vector field X of $T(JE)$ written as

$$X = a^B\frac{\partial}{\partial x^B} + b^r\frac{\partial}{\partial v^r} + c_B^r\frac{\partial}{\partial v_B^r}$$

we calculate $i_X\,\mathrm{d}\Theta$:

$$\begin{aligned} i_X\,\mathrm{d}\Theta &= b^r\left(\frac{\partial\mathcal{L}}{\partial v^r} - \mathrm{d}Q_r\right) + c_A^r\left(\frac{\partial\mathcal{L}}{\partial v_A^r} - (-1)^{A(A+r)}\,\mathrm{d}x^A\,Q_r\right)\\ &\quad + a^B(v_B^r\,\mathrm{d}Q_r + \mathrm{d}v_A^r\,Q_r) - \mathrm{d}v^r\frac{\partial}{\partial v^r}\left(a^B i_B\mathcal{L}\right) - \mathrm{d}v_A^r\frac{\partial}{\partial v_A^r}\left(a^B i_B\mathcal{L}\right)\\ &\quad + \mathrm{d}x^A\mathrm{d}v_A^r\left(a^B i_B Q_r\right) + \left(\mathrm{d}v^r - dx^A v_A^r\right) i_X\,\mathrm{d}Q_r. \end{aligned} \tag{13}$$

Consider now a local injection i and calculate

$$i^\star\cdot js^\star\cdot i_X\,\mathrm{d}\Theta = 0. \tag{14}$$

We observe that the terms multiplying the coefficients C_A^r vanish only if $\mathcal{L}$ is geometrical and $Q_r = \mathcal{L}/v'^r$. Moreover, the terms multiplying a^B cannot be zero in general. Our final prescription for the Poincaré–Cartan form is hence

$$\Theta = \mathcal{L} + \left(\mathrm{d}v^r - \mathrm{d}x^A v_A^r\right)\ \frac{\partial\mathcal{L}}{\partial.}v'^r \tag{15}$$

We can finally state the following

Proposition 3.2. Let $\mathcal{L}$ be a geometrical Lagrangian form; the expression

$$i^\star \cdot js^\star \cdot i_X \, \mathrm{d}\Theta = 0 \tag{16}$$

is zero for each vertical vector field $X \in T(JE)$ iff the following equations are satisfied:

$$\frac{\partial \mathcal{L}}{\partial v^r} - \mathrm{d}\frac{\partial \mathcal{L}}{\partial v'^r} = 0. \tag{17}$$

4. Symmetries, superdiffeomorphisms and jet extensions

We assume that the configuration bundle $E \to M$ is the fibred product (see, e.g., [34]) of the bundle $V(M)$ of linear coframes on M, the bundle of connections $C(M)$ on some principal superfibre bundle $P(M, G)$ and a further bundle $F(M)$ of zero forms with values in some vector superspace $\mathbf{F}$. We denote by g the super Lie (SL) algebra of G. More precisely,

$$E = V(M) \times_M C(M) \times_M F(M)$$

and a local section s of E is $s(x^A) = (x^A, e^B = \mathrm{d}x^C e_C{}^B(x), \omega^a = \mathrm{d}x^C \omega_C^a(x), \xi^r(x))$.

A point $u \in E$ describes a connection, an extra field and a metric, by regarding e^B as a Lorenzian coframe; in other words, the metric is $\eta_{AB}\, e^A e^B$.

Every element a of G naturally acts on $C(M)$ and, by means of suitable representations, on $V(M)$ and $F(M)$.

We denote by C_{bc}^a the structure constants of the SL algebra g and by the symbols ρ_{aC}^A, τ_{as}^r the generators of the representations of the actions of g on TM and F respectively.

By considering a fibre-depending action of G on E, one obtains a vertical motion in E the tangent vector field of which will be denoted by Y. Then the following proposition holds.

Proposition 4.1. Let us denote by A^a an element of g; the local representation of the vertical field Y describing the vertical motion induced by A^a is

$$Y = (D_B A^a)\frac{\partial}{\partial \omega_A^a} - e_B{}^C A^a \rho_{aC}^E \frac{\partial}{\partial e_B{}^E} - \xi^r A^a \tau_{ar}^s \frac{\partial}{\partial \xi^s} \tag{18}$$

where

$$(D_B A^a) = \frac{\partial A^a}{\partial x^B} + (-1)^{Bd} A^d \omega_B^c C_{cd}^a.$$

The bundle E is a bundle of (super)-geometrical quantities; by considering a superdiffeomorphism $\beta: M \to M$, one obtains a natural action on E given by the pull-back $\hat{\phi} = \beta^\star$. Accordingly, the following holds.

Proposition 4.2. Let β_t be a one-parameter SL group of SA diffeomorphisms of the supermanifold M. Then the couple $(\beta_t, \hat{\phi}_t = \beta_t^{-1\,\star})$ is a one-parameter SL group of SA automorphisms of E. For any local section s the map $s_t = \hat{\phi}^{-1} \circ s \circ \beta_t$ is also a section.

The action of $\hat{\phi}_t$ can be described as follows. If $u = (x^A, e^B = \mathrm{d}x^A e_A{}^B, \omega^a = \mathrm{d}x^A \omega_A{}^a, \xi^r)$ is a point of E and $v = \dot{\beta}_t$, then the coordinate description of $u \to w = \hat{\phi}_t(u)$ is

$$x^A \to x^A + tv^A \qquad e_A{}^B \to e_A^B - tv^C_{,A}\, e_C{}^B \qquad \omega_A^a \to \omega_A^a - tv^C_{,A}\omega_C^a \qquad \xi^r \to \xi^r.$$

Now, there is a large arbitrariness involved in the choice of the coordinates in E. In particular, we use the coframe e^A as basis for the connection one forms and set $\omega^a = e^B \tilde{\omega}_B{}^a$.

Of course, $\omega_A^{\ a} = e_A^{\ B}\tilde{\omega}_B^{\ a}$. In this way the coordinate description of the map $u \to w = \hat{\phi}_t(u)$ is

$$x^A \to x^A + tv^A \qquad e_A^{\ B} \to e_A^B - tv^C_{,A}\, e_C^{\ B} \qquad \tilde{\omega}_A^{\ a} \to \tilde{\omega}_A^{\ a} \qquad \xi^r \to \xi^r. \tag{19}$$

Under the above choice, the following proposition holds.

Proposition 4.3. The vector field Z, tangent to $\hat{\phi}_t$, has the following representation in the coordinates $x^A, e_A^{\ B}, \tilde{\omega}_A^{\ a}, \xi^r$

$$Z = v^A \frac{\partial}{\partial x^A} - \left(\frac{\partial v^C}{\partial x^B}\right) e_C^{\ D} \frac{\partial}{\partial e_B^{\ D}}. \tag{20}$$

Let us now consider the jet extension $j(\hat{\phi}_t)$ of $\hat{\phi}_t$ to JE. The couple $(\beta_t, j(\hat{\phi}_t))$ is still an automorphism of JE. We introduce the following 'covariant coordinates', for which the relation to the standard ones on a section is

$$\begin{aligned} js^\star\Omega_{AB}^{\ a} &= -2\frac{\partial \omega_B^{\ a}}{\partial x^A} - (-1)^{A(B+b)}\omega_B^{\ b} \wedge \omega_A^{\ c} C_{cb}^{\ a} \\ js^\star T_{AB}^{\ C} &= -2\frac{\partial e_B^{\ C}}{\partial x^A} - (-1)^{A(B+D)} e_B^{\ D} \wedge \omega_A^{\ a} \rho_{aD}^{\ C} \\ js^\star \Xi_A^r &= \frac{\partial \xi^r}{\partial x^A} + (-1)^{At}\xi^t \omega_A^{\ a} \tau_{at}^r. \end{aligned} \tag{21}$$

Now, as the last step, we use again 'covariant coframes' (CC) coordinates:

$$\begin{aligned} &\tilde{e}_A^{\ B} = e_A^{\ B} && \tilde{\omega}_A^{\ a} = e_A^{-1\ S}\, \omega_S^{\ a} \\ &\tilde{\xi}^r = \xi^r && \tilde{T}_{AB}^{\ C} = e_B^{-1\ S}\, e_A^{-1\ R} T_{RS}^{\ C}(-1)^{A(S+B)} \\ &\tilde{\Xi}_A^{\ r} = e_A^{-1\ S}\, \Xi_S^{\ r} && \tilde{\Omega}_{AB}^{\ C} = e_B^{-1\ S}\, e_A^{-1\ R}\Omega_{RS}^{\ C}(-1)^{A(S+B)}. \end{aligned} \tag{22}$$

We now consider the lift of Z to JE. The following proposition holds.

Proposition 4.4. The expression of jZ in local CC coordinates

$$(x^A, e_A^{\ D}, \tilde{\omega}_A^{\ a}, \xi^r, \tilde{T}_{KP}^{\ C}, \tilde{\Omega}_{KP}^{\ a}, \tilde{\Xi}_A^{\ r})$$

is

$$\begin{aligned} jZ = v^A \frac{\partial}{\partial x^A} &- \left(\frac{\partial v^C}{\partial x^B}\right) e_C^{\ D}\, \frac{\partial}{\partial e_B^{\ D}} + 2e_P^{-1\ A} e_K^{-1\ B} \frac{\partial^2 v^J}{\partial x^B \partial x^A}\, e_J^{-1\ T} \tilde{\omega}_T^{\ a}\, (-1)^{K(A+P)} \frac{\partial}{\partial \tilde{\Omega}_{KP}^{\ a}} \\ &+ 2e_P^{-1\ A}\, e_K^{-1\ B}\, \frac{\partial^2 v^J}{\partial x^B \partial x^A}\, e_J^{\ C}\, (-1)^{K(A+P)}\, \frac{\partial}{\partial \tilde{T}_{KP}^{\ C}}. \end{aligned} \tag{23}$$

Let us return to consider the local section s_t. The pull-back of its jet extension

$$js_t = j(\hat{\phi}_t^{-1}) \circ js \circ \beta_t \tag{24}$$

can be regarded as a way in which to describe the action of superdiffeomorphisms on JE.

To this end we compute the action of its pull-back on $\mathcal{L} \in \mathrm{hor}\,\Omega^m JE$

$$js_t^\star \mathcal{L} = \beta_t^\star \circ js^\star \circ j(\hat{\phi}_t^{-1})^\star \mathcal{L}. \tag{25}$$

Differentiating with respect to t and putting $t = 0$ we get

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (js_t)^\star \mathcal{L} = L_v (js)^\star \mathcal{L} - (js)^\star L_{jZ}\mathcal{L}. \tag{26}$$

This suggests to us the following definition.

Definition 4.1. Let z be a point of JE and set $z' = j(\hat{\phi}_t)(z)$. $\mathcal{L}$ is invariant with respect to superdiffeomorphisms if

$$\big((j\hat{\phi}_t^{-1})^*\mathcal{L}(z)\big)(z') = \mathcal{L}(z'). \tag{27}$$

The infinitesimal version of this requirement can be explicitly written without any reference to sections; it is

$$L_{jZ}\,\mathcal{L} = 0. \tag{28}$$

Before proceeding further, we shall derive a useful relation. To this end, let us consider a section s and compute $(js)^* L_{jZ}\mathcal{L}$. We have

$$\begin{aligned}(js)^* L_{jZ}\mathcal{L} &= (js)^*(jZ \rfloor \mathrm{d}\mathcal{L}) + \mathrm{d}(js)^*(jZ \rfloor \mathcal{L})\\ &= (js)^*[A \rfloor \mathrm{d}\mathcal{L} + (js)_*(v) \rfloor \mathrm{d}\mathcal{L}] + \mathrm{d}(js)^*(jZ \rfloor \mathcal{L})\\ &= (js)^*[A \rfloor \mathrm{d}\mathcal{L}] + L_v (js)^*\mathcal{L}\end{aligned} \tag{29}$$

where $A = jZ - (js)_*(v)$ is a vector of $T_{js(x)}(JE)$.

We can now formulate the following proposition:

Proposition 4.5. Let us fix a section $s(x)$; the vector A has the following coordinate representation:

$$\begin{aligned}A = &-(L_v e^B(x))_C \frac{\partial}{\partial e_C^{\,E}} - (L_v \omega^b(x))_C \frac{\partial}{\partial \omega_C^{\,b}} - (L_v \xi^r(x))\,\frac{\partial}{\partial \xi^r}\\ &+(L_v\,\mathrm{d}e^B(x))_{CD}\frac{\partial}{\partial e_{C,D}^{\,E}} + (L_v\,\mathrm{d}\omega^b(x))_{CD}\,\frac{\partial}{\partial \omega_{C,D}^{\,b}} - (L_v\,\mathrm{d}\xi^r(x))_C \frac{\partial}{\partial \xi_C^{\,r}}\\ &-\frac{\partial^2 v^j}{\partial x^B \partial x^A} e_j^{\,T}\,\frac{\partial}{\partial e_{A,B}^{\,T}} - \frac{\partial^2 v^j}{\partial x^B \partial x^A}\omega_j^{\,a}\frac{\partial}{\partial \omega_{A,B}^{\,a}}.\end{aligned} \tag{30}$$

Having established this, we find the following result about the relations between Lie derivatives in JE and M.

Proposition 4.6. The following relation holds:

$$(js)^* L_{jZ}\mathcal{L} - L_v (js)^*\mathcal{L} = -\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}(js_t)^*\mathcal{L} = (js)^*[A \rfloor \mathrm{d}\mathcal{L}]. \tag{31}$$

The Lagrangian form $\mathcal{L}$ can hence be written in the following form:

$$\mathcal{L} = \frac{1}{n!}\sum_{A_1\ldots A_n} \mathrm{d}x^{A_1}\ldots \mathrm{d}x^{A_n}\mathcal{L}_{A_1\ldots A_n}(x^A, e_A^{\,D}, \omega_A^{\,a}, \xi^r, T_{KP}^{\,C}, \Omega_{KP}^{\,a}, \Xi_A^{\,r}) \tag{32}$$

or, by using CC coordinates, in the equivalent form

$$\mathcal{L} = \frac{1}{n!}\sum_{A_1\ldots A_n} e^{A_1}\ldots e^{A_n}\tilde{\mathcal{L}}_{A_1\ldots A_n}(x^A, e_A^{\,D}, \tilde{\omega}_A^{\,a}, \xi^r, \tilde{T}_{KP}^{\,C}, \tilde{\Omega}_{KP}^{\,a}, \tilde{\Xi}_A^{\,r}). \tag{33}$$

We are now ready to compute $L_{jZ}\mathcal{L}$. By remarking that $L_{jZ}e^A = 0$, we get the following result.

Proposition 4.7. Let us choose CC coordinates in JE. Then the Lagrangian $\mathcal{L}$ is invariant with respect to superdiffeomorphisms if and only if

$$jZ(\tilde{\mathcal{L}}_{A_1\ldots A_n}) = 0. \tag{34}$$

5. The Lie group action and superdiffeomorphisms

Now we return to consider the action of G on E; since this action can be lifted to JE, we consider the first jet extension of the field $Y \in \mathrm{vert}(E)$. We shall say that $\mathcal{L}$ is G-invariant if

$$L_{jY}\mathcal{L} = 0. \tag{35}$$

Clearly a more general definition of invariance, including divergence terms, can also be proposed; however, the present one is sufficient for our purposes.

The following proposition holds.

Proposition 5.1. Let us choose standard coordinates in JE. The local expression of jY is

$$\begin{aligned} jY = {} & (D_B A^a)\frac{\partial}{\partial \omega_A^{\ a}} - e_B^{\ C} A^a \rho_{aC}^{\ E} \frac{\partial}{\partial e_B^{\ E}} - \xi^r A^a \tau_{ar}^{\ s} \frac{\partial}{\partial \xi^s} \\ & -\big[2D_B D_A A^a - \big((-1)^{B(A+b)}(D_A A^b)\omega_B^{\ d} + (-1)^{bA}(D_B A^b)\omega_A^{\ d}\big)C_{db}^{\ a}\big]\frac{\partial}{\partial \Omega_{BA}^{\ c}} \\ & -\Xi_B^{\ r} A^b \tau_{br}^{\ s}\frac{\partial}{\partial \Xi_B^{\ s}} - T_{BA}^{\ C} A^b \rho_{bC}^{\ E} \frac{\partial}{\partial T_{BA}^{\ E}}. \end{aligned} \tag{36}$$

Let us then consider equation (35). In this equation there are terms multiplying the second and the first derivatives of A^a. Requiring that these terms vanish implies

$$\frac{\partial \mathcal{L}}{\partial \Omega_{AB}^{\ a}} = (-1)^{AB}\frac{\partial \mathcal{L}}{\partial \Omega_{BA}^{\ a}} \qquad \frac{\partial \mathcal{L}}{\partial \omega_A^{\ a}} = 0. \tag{37}$$

In other words, $\mathcal{L}$ has to depend on the first derivatives of ω only through the curvature components and, moreover, $\mathcal{L}$ has to be independent of ω. On the other hand, the requirement that linear terms in A^a vanish, implies the following equations in standard and CC coordinates, respectively:

$$\begin{aligned} & e_B^{\ C} A^a \rho_{aC}^{\ E}\frac{\partial \mathcal{L}}{\partial e_B^{\ E}} + \xi^r A^a \tau_{ar}^{\ s}\frac{\partial \mathcal{L}}{\partial \xi^s} + \Xi_B^{\ r} A^a \tau_{ar}^{\ s}\frac{\partial \mathcal{L}}{\partial \Xi_B^{\ s}} \\ & \qquad + T_{BA}^{\ C} A^a \rho_{aC}^{\ E}\frac{\partial \mathcal{L}}{\partial T_{BA}^{\ E}} + \Omega_{BA} A^a C_{ac}^{\ b}\frac{\partial \mathcal{L}}{\partial \Omega_{BA}^{\ b}} = 0 \\ & e_B^{\ C} A^a \rho_{aC}^{\ E}\frac{\partial \mathcal{L}}{\partial e_B^{\ E}} + \xi^r A^a \tau_{ar}^{\ s}\frac{\partial \mathcal{L}}{\partial \xi^s} + \big(\tilde{\Xi}_P^{\ r} A^a \tau_{ar}^{\ d} - A^a \rho_{ap}^{\ E}\tilde{\Xi}_E^{\ d}\big)\frac{\partial \mathcal{L}}{\partial \tilde{\Xi}_P^{\ d}} \\ & \qquad + \big(\tilde{T}_{KP}^{\ C} A^a \rho_{aC}^{\ E} - 2A^a \rho_{aK}^{\ C}\tilde{T}_{CP}^{\ E}\big)\frac{\partial \mathcal{L}}{\partial \tilde{T}_{KP}^{\ E}} \\ & \qquad + \big(\tilde{\Omega}_{KP}^{\ c} A^a C_{ac}^{\ e} - 2A^a \rho_{aK}^{\ C}\tilde{\Omega}_{CP}^{\ e}\big)\frac{\partial \mathcal{L}}{\partial \tilde{\Omega}_{KP}^{\ e}} = 0. \end{aligned} \tag{38}$$

Let us summarize our general discussion. The Lagrangian $\mathcal{L}$ is invariant with respect to the combined actions of G and of superdiffeomorphisms if

(i) $\mathcal{L}/x^A = 0$ and $\mathcal{L}/\omega_A^{\ a} = 0$;

(ii) $\mathcal{L}$ depends on the derivatives of $e_A^{\ b}$ and $\omega_A^{\ a}$ only through of the components the curvature and torsion forms;

(iii) equation (39) holds;

(iv) $\tilde{\mathcal{L}}_{A_1 \ldots A_n}/e_A^{\ B} = 0$.

6. The Poincaré–Cartan form

We are finally ready to discuss the Poincaré–Cartan form Θ; from proposition 3.2 we know that we have to assume that that $\mathcal{L}$ is a geometrical Lagrangian form. In the opposite case no classical form analogous to Poincaré–Cartan exists.

First of all, we analyse the structure of Lagrange field equations. In general, they have the form

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial\xi^r} &= (-1)^{Cr}\frac{\mathrm{d}}{\mathrm{d}x^C}\frac{\partial\mathcal{L}}{\partial\xi^r_{,C}} \\ \frac{\partial\mathcal{L}}{\partial e_B^{\ A}} &= (-1)^{C(B+A)}\frac{\mathrm{d}}{\mathrm{d}x^C}\frac{\partial\mathcal{L}}{\partial e_{B,C}^{\ A}} \\ \frac{\partial\mathcal{L}}{\partial\omega_B^{\ a}} &= (-1)^{C(B+a)}\frac{\mathrm{d}}{\mathrm{d}x^C}\frac{\partial\mathcal{L}}{\partial\omega_{B,C}^{\ a}}. \end{aligned} \tag{39}$$

Now, by using covariant coordinates, we write the field equations in the form

$$\begin{aligned} &\frac{\partial\mathcal{L}}{\partial\xi^r} - (-1)^{rA}\frac{\mathrm{d}}{\mathrm{d}x^A}\frac{\partial\mathcal{L}}{\partial\Xi_A^{\ r}} + (-1)^{rA}\omega_A^{\ b}\tau_{br}^{\ \ s}\frac{\partial\mathcal{L}}{\partial\Xi_A^{\ s}} = 0 \\ &\frac{\partial\mathcal{L}}{\partial e_M^{\ A}} - 2(-1)^{C(M+A)}\omega_C^{\ a}\rho_{aA}^{\ \ B}\frac{\partial\mathcal{L}}{\partial T_{SM}^{\ \ B}} + 2(-1)^{B(M+A)}\frac{\mathrm{d}}{\mathrm{d}x^B}\frac{\partial\mathcal{L}}{\partial T_{BM}^{\ \ A}} = 0 \\ &\frac{\partial\mathcal{L}}{\partial\omega_M^{\ a}} - 2(-1)^{a(C+D)}e_C^{\ D}\rho_{aD}^{\ \ B}\frac{\partial\mathcal{L}}{\partial T_{MC}^{\ \ B}} - 2(-1)^{D(M+A)}\omega_D^{\ d}C_{da}^{\ \ b}\frac{\partial\mathcal{L}}{\partial\Omega_{CM}^{\ \ b}} \\ &\qquad + 2(-1)^{B(M+A)}\frac{d}{dx^B}\frac{\partial\mathcal{L}}{\partial\Omega_{BM}^{\ \ a}} = 0. \end{aligned} \tag{40}$$

Finally, by relying on the assumption that the Lagrangian is 'geometrical', we can write the field equations in a more concise and better suited form.

First of all, let us introduce the forms

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial\omega_A^{\ a}} &= \mathrm{d}x^A\frac{\partial\mathcal{L}}{\partial\omega^a}(-1)^{A(A+a)} \\ \frac{\partial\mathcal{L}}{\partial\Omega_{AB}^{\ \ a}} &= \mathrm{d}x^B\mathrm{d}x^A\frac{\partial\mathcal{L}}{\partial\Omega^a}(-1)^{(A+B)(A+B+a)} \end{aligned} \tag{41}$$

and, similarly, $\mathcal{L}/\xi^r, \mathcal{L}/e^A, \mathcal{L}/\Xi^r, \mathcal{L}/T^A$.

Then, the following theorem holds true:

Theorem 6.1. Let us assume that $\mathcal{L}$ is a geometric Lagrangian. The field equations can be written in the form

$$\begin{aligned} G_r(\xi) &\equiv \frac{\partial\mathcal{L}}{\partial\xi^r} - \mathrm{D}\frac{\partial\mathcal{L}}{\partial\Xi^r} = 0 \\ G_A(e) &\equiv \frac{\partial\mathcal{L}}{\partial e^A} + \mathrm{D}\frac{\partial\mathcal{L}}{\partial T^A} = 0 \\ G_a(\omega) &\equiv \frac{\partial\mathcal{L}}{\partial\omega^a} + \mathrm{D}\frac{\partial\mathcal{L}}{\partial\Omega^a} + J_a = 0 \end{aligned} \tag{42}$$

where the 'current density' J_a is

$$J_a = (-1)^{aD}e^D\rho_{aD}^{\ \ B}\frac{\partial\mathcal{L}}{\partial T^B} + (-1)^{as}\xi^s\tau_{as}^{\ \ t}\frac{\partial\mathcal{L}}{\partial\Xi^t}. \tag{43}$$

The symbol D denotes the exterior covariant derivative operator.

Now we enter into details about the Poincaré–Cartan form. We put

$$\Theta = \mathcal{L} - (\mathrm{D}e^A - T^A)Q_A(e) - (\mathrm{D}\omega^a - \Omega^a)Q_a(\omega) - (\mathrm{D}\xi^r - \Xi^r)Q_r(\xi) \tag{44}$$

for some $Q_r(\xi) \in \mathrm{hor}\,\Omega^{m-1}JE$ and some $Q_a(\omega)$, $Q_A(e) \in \mathrm{hor}\,\Omega^{m-2}JE$. Thence:

Theorem 6.2. Let us assume that $\mathcal{L}$ is geometric and consider a vertical field $X \in T(JE)$. If $Q_r(\xi) = \mathcal{L}/\Xi^r$, $Q_a(\omega) = \mathcal{L}/\Omega^a$ and $Q_A(e) = \mathcal{L}/T^A$, the equation $i_X\,\mathrm{d}\Theta = 0$ is verified iff the field equations (42) for $\mathcal{L}$ hold.

To derive symmetry conditions we now compute $L_{jZ}\Theta$. Using the expression for jZ we get

$$L_{jZ}e^A = L_{jZ}T^A = L_{jZ}\Omega^a = L_{jZ}\Xi^r = 0. \tag{45}$$

Consequently,

$$L_{jZ}\Theta = L_{jZ}\mathcal{L} + (\mathrm{D}e^A - T^A)L_{jZ}Q_A(e)(\mathrm{D}\omega^a - \Omega^a)L_{jZ}Q_a(\omega) + (\mathrm{D}\xi^r - \Xi^r)L_{jZ}Q_r(\xi). \tag{46}$$

This last equation gives the required relation between the symmetries of Lagrangian dynamics and the Poincaré–Cartan formalism for superfield theory.

7. An example. The Poincaré–Cartan form of $N = 1$ supergravity

We consider as an example the theory of $N = 1$ supergravity. The supermanifold M has dimension $(4, 4)$ and the indices i, j, k, l run through $1, \ldots, 4$. We denote by $e^i = \mathrm{d}x^A e_A^{\ i}$ (i even), by $e = \mathrm{d}x^A e_A^{\alpha}$ (α odd) and by $\theta = \mathrm{D}e$. Following [35] we consider the following supermanifold Lagrangian 4-form:

$$\mathcal{L} = \varepsilon_{ijkl}e^i e^j \Omega^{kl} + 4\theta C\gamma_i\gamma_5 e e^i \tag{47}$$

which is manifestly geometric. No auxiliary field is present. We have

$$\begin{aligned}
&\frac{\partial\mathcal{L}}{\partial e^i} = 2e^j\Omega^{ks}\varepsilon_{ijks} - 4\theta C\gamma_i\gamma_5 e e^i \qquad \frac{\partial\mathcal{L}}{\partial T^i} = 0\\
&\frac{\partial\mathcal{L}}{\partial e} = -4(\theta C\gamma_i\gamma_5)^T e^i \qquad \frac{\partial\mathcal{L}}{\partial\theta} = 4C\gamma_i\gamma_5 e e^i\\
&\frac{\partial\mathcal{L}}{\partial\omega^{ik}} = 0 \qquad \frac{\partial\mathcal{L}}{\partial\Omega^{ks}} = e^i e^j\varepsilon_{ijks}
\end{aligned} \tag{48}$$

and the Poincaré–Cartan form is given by

$$\Theta = 2\varepsilon_{ijkl}e^i e^j\Omega^{kl} + 8\theta C\gamma_i\gamma_5 e e^i - \mathrm{D}\omega^{ik}\varepsilon_{ikps}e^p e^s - 4\mathrm{D}eC\gamma_i\gamma_5 e e^i. \tag{49}$$

An analogous procedure can also be followed with minor changes for $N = 2$ supergravity theory, but the formulae become more complicated.

Acknowledgments

This work was carried out under the auspices of the GNFM of CNR and under the research program 'Metodi Geometrici e Probabilistici in Fisica Matematica' of MURST; one of the authors (TV) would also like to thank the GNFM for a visiting professorship when this work was undergone.

References

[1] Wess J and Bagger J 1983 *Supersymmetry and Supergravity* (Princeton NJ: Princeton University Press)
[2] Schwarz J H Green M B and Witten E 1987 *Superstring Theory* (Cambridge: Cambridge University Press)
[3] Ferrara S and Van Nieuwenhuizen P 1976 Consistent supergravity with complex spin-3/2 gauge field *Phys. Rev. Lett.* 1669.
[4] Cianci R 1990 *Introduction to Supermanifolds* (Napoli: Bibliopolis)
[5] Berezin F A 1966 *The Methods of Second Quantization* (New York: Academic)
[6] Berezin F A and Leites D A 1975 Supermanifolds *Sov. Math. Dokl.* **16** 1218
[7] De Witt B 1984 *Supermanifolds* (Cambridge: Cambridge University Press)
[8] Bruzzo U, Bartocci C and Hernández Ruipérez D 1991 *The Geometry of Supermanifolds* (Dordrecht: Kluwer)
[9] Henneaux M 1988 *Classical Foundations of* BRST *Symmetry. Monograph and Textbooks in Physical Science. Lecture Notes 7* (Napoli: Bibliopolis)
[10] Francaviglia M, Ferraris M and Volovich I 1993 BRST-invariance and Poincaré–Cartan forms to appear
[11] Henneaux M 1991 Spacetime locality of the brst formalism *Commun. Math. Phys.* **140** 1
[12] Arnold V 1969 *Methódes Methématiques de la Mécanique Classique* (Moscow: MIR)
[13] Vladimirov V S and Volovich I V 1984 Superanalysis. i. differential calculus *Theor. Math. Phys.* **59** 317
[14] Volovich I V, Vladimirov V S and Zelenov E I 1992 *p-adic Numbers in Mathematical Physics* (Singapore: World Scientific)
[15] Khrennikov A Yu 1990 Mathematical methods of non-Archimedean physics *Russian Math. Survey* **45** 87
[16] Bruzzo U and Cianci R 1987 Variational calculus on supermanifolds and invariance properties of superspace field theories *J. Math. Phys.* **28** 786
[17] Francaviglia M and Ferraris M 1991 The Lagrangian approach to conserved quantities in general relativity in: *Mechanics, Analysis and Geometry; 200 years after Lagranges* ed M Francaviglia (Amsterdam: North-Holland)
[18] Vladimirov V S and Volovich I V 1985 Superanalysis. ii. integral calculus *Theor. Math. Phys.* **60** 743
[19] Rogers A 1980 A global theory of supermanifolds *J. Math. Phys.* **21** 1352
[20] Rogers A 1981 Super Lie groups: global topology and local structure *J. Math. Phys.* **22** 939
[21] Rothstein M 1986 The axioms of supermanifolds and a new structure arising from them *Trans. Amer. Math. Soc.* **297** 159
[22] Jadczyk A and Pilch K 1981 Superspaces and supersymmetries *Commun. Math. Phys.* **78** 373
[23] Cianci R and Khrennikov A 1993 p-adic superanalysis i: infinitely-generated Banach superalgebras and algebraic groups *J. Math. Phys.* 34 1986
[24] Cianci R and Khrennikov A 1993 p-adic superanalysis ii: supermanifolds *J. Math. Phys.* 34 1995
[25] Cianci R and Khrennikov A 1994 p-adic superanalysis iii: the structure of p-adic super Lie groups *J. Math. Phys.*
[26] Bourbaki N 1967 *Variétés Différentielles at Analytiques* (Paris: Hermann)
[27] Khrennikov A Yu 1988 Functional superanalysis. *Russian Math. Surv.* 43 103
[28] Serre J P 1965 *Lie Algebras and Lie Groups* (New York: W A Benjamin)
[29] Reina C, Catenacci R and Teofilatto P 1985 On the body of supermanifolds *J. Math. Phys.* 26 671
[30] Hernandez Ruiperez D and Munoz Masque J 1984 Global variational calculus on graded manifolds: i – graded jet-bundles *J. Math. Pures Appl.* 63
[31] Rogers A 1985 On the existence of global integral forms on supermanifolds *J. Math. Phys.* 26 2749
[32] Rogers A 1985 Realising the berezin integral as a superspace contour integral *J. Math. Phys.* 27 710
[33] Wess J and Zumino B 1977 Superspace formulation of supergravity. *Phys. Lett.* **B66** 361
[34] Bott R and Tu L W 1982 *Differential Forms in Algebraic Topology* (Berlin: Springer)
[35] Ne'eman Y and Regge T 1978 Gauge theory of gravity and supergravity on a group manifold *Rivista Nuovo Cim.* 1